Grade 6 Math Circles
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Mathematical Logic

I want to acknowledge that I created many parts of this lesson based on *How to Prove It* by Daniel J. Velleman.

Motivation

How do we prove a mathematical statement? The answer is: we use a method called *deductive reasoning*.

Stop and Think

Define *reasoning* in your own words:

Some of you might say, "Okay... I know what *reasoning* is, but what does *deductive* mean?" We will first take a look at an example of deductive reasoning.

Example 1

1. I will go to work either tomorrow or today (or both).
2. I am going to stay home today.
3. Therefore, I will go to work tomorrow.

In the above example, we arrived at a *conclusion* that I will go to work tomorrow from the assumption that the first two statements, called *premises*, are true. The conclusion seems to be forced on us somehow by the premises.

But is this conclusion really correct? After all, isn't it possible that I will stay home today, and then wake up with COVID-19 tomorrow and end up staying home again? Notice that in this case, the first premise, "I will go to work either tomorrow or today", would be false as well. Although we cannot guarantee that the conclusion is true, it can ONLY be false if at least one of the premises is also false. In other words, if all the premises are true, then the conclusion must be true as well. This is the standard we will use to judge the correctness of deductive reasoning. We will say that an argument is *valid* when if the premises are all true, then it must be the case that the conclusion is true as well. Example 1 is a valid argument.

Example 2

Here is an example of an INVALID deductive argument:

1. Either Alex is guilty or Claire is guilty.
2. Either Claire is guilty or Min is guilty.
3. Therefore, either Alex is guilty or Min is guilty.

The argument is invalid because the conclusion could be false even if both premises are true. If Claire were guilty, but Alex and Min were both innocent, then both premises would be true and the conclusion would be false.

We can learn something about what makes an argument valid by referring back to Example 1. It introduces two possibilities in the first premise, rules out the second possibility with the second premise, and then concludes that the first possibility must be the case. In other words, it has the form:

Example 1 Continued

1. P or Q.
2. Not Q.
3. Therefore, P.

It is this form, and not the subject matter, that makes this argument valid.

Stop and Think

What English sentences do P and Q represent in Example 1?

- P: I will go to work tomorrow.
- Q: I will go to work today.

Replacing certain statements in each argument with letters like P and Q has an advantage. It keeps us from being distracted by the language itself. Hence, we will continue to use letters to represent statements, but only for statements that are either true or false. For example, questions and exclamations will not be allowed.

Logical Connectives

We will now focus on words that are used to combine statements to form more complex statements. Just like representing statements with letters, it is also useful to use symbols, called *connectives* or *connective symbols*, to stand for some of the words used to combine statements. Here are our first three connectives and what they mean:

symbol	meaning
$\wedge$	and
$\vee$	or
$\neg$	not

Thus, if P and Q stand for two statements, then we will write $P \vee Q$ to stand for the statement "P or Q", $P \wedge Q$ for "P and Q", and $\neg P$ for "not P" or "P is false." The statement $P \vee Q$ is sometimes called the *disjunction* of P and Q, $P \wedge Q$ is called the *conjunction* of P and Q, and $\neg P$ is called the *negation* of P.

Exercise 1

What English sentences are represented by the following expressions? (FYI, Toto is the writer's dog! He is a small poodle, 10 years old!)

1. $(\neg S \wedge L) \vee S$, where S stands for "Toto is sleepy" and L stands for "Toto is lazy."
2. $\neg S \wedge (L \vee S)$, where S and L have the same meanings as 1.
3. $\neg(S \wedge L) \vee S$, with S and L still as before.

Exercise 1 Solution

1. Either Toto isn't sleepy and he is lazy, or he is sleepy.
2. Toto isn't sleepy, and either he is lazy or he is sleepy. (Notice how the placement of the

word *either* in English changes according to where the parentheses are.)

3. Either Toto isn't both sleepy and lazy, or Toto is sleepy. (The word *both* in English also helps distinguish the different possible positions of parentheses.)

It is important to keep in mind that the symbols $\wedge, \vee$, and $\neg$ don't correspond to all uses of the words *and, or,* and *not* in English. For example, the symbol $\wedge$ cannot be used to represent the use of the word *and* in the sentence "Alex and Min are friends," because the word *and* in this sentence is not being used to combine two statements. On that note, here are two rules to keep in mind:

Rule 1

1. The symbols $\wedge$ and $\vee$ can only be used *between two statements*, to form their conjunction or disjunction.
2. The symbol $\neg$ can only be used *before a statement*, to negate it.

Here is a helpful analogy to understand the above rules:

Analogy of Rule 1

This is an analogy involving your algebra knowledge!

1. Symbols $+, -, \cdot$, and $\div$ can be used *between two numbers*, as operators.
2. The symbol $-$ can also be used *before a number*, to negate it.

Truth Values & Truth Tables

When we evaluate the truth or falsity of a statement, we assign one of the labels *true* or *false* to it, and this label is called its *truth value*. We can think about how the word *and* contributes to the truth value of a statement connected by it. A statement of the form $P \wedge Q$ can only be true if both P and Q are true; if either P or Q is false, then $P \wedge Q$ will be false as well. Because P and Q are both statements that are either true or false, we can summarize all the possibilities with the table below:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

This is called a *truth table* for the statement $P \wedge Q$.

The truth table for $\neg P$ is also quite easy to construct because it is just the negation of the truth value of P.

P	$\neg P$
T	F
F	T

The truth table for $P \vee Q$ requires a little more brain power. With a little thought, you can convince yourself that the last three lines should turn out as below:

P	Q	$P \vee Q$
T	T	?
T	F	T
F	T	T
F	F	F

Let's give our attention to the case where both P and Q are true. Should $P \vee Q$ be true or false in this case? In other words, does $P \vee Q$ mean "P or Q, or both" (*inclusive or*) or does it mean "P or Q, but not both" (*exclusive or*)? In English, we usually mean **exclusive or** when we say *or*. In mathematics, we always mean **inclusive or** when we say *or*, unless specified otherwise. Hence, we will interpret $\vee$ as inclusive or. Now, we can complete the above table.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Using the rules summarized in these truth tables, we can now work out truth tables for more complex formulas. We first work out the truth values of the component of a formula, starting with the individual letters and working up to more complex formulas one step at a time.

Exercise 2

Make a truth table for the formula $\neg(P \vee \neg Q)$.

Exercise 2 Solution

P	Q	$\neg Q$	$P \vee \neg Q$	$\neg(P \vee \neg Q)$
T	T	F	T	F
T	F	T	T	F
F	T	F	F	T
F	F	T	T	F

Finally, we will convince ourselves that Example 1 is valid using what we have learned so far. We can represent the argument in Example 1 symbolically as follows:

Example 1 Continued

1. $P \vee Q$
2. $\neg Q$
3. $\therefore P$ (The symbol $\therefore$ means *therefore*.)

We can create the following truth table:

P	Q	$P \vee Q$	$\neg Q$
T	T	**T**	F
T	F	**T**	**T**
F	T	**T**	F
F	F	F	**T**

Recall that we say that an argument is *valid* when if the premises are all true, then it must be the case that the conclusion is true. Notice that the second line is the only case where both premises,

$P \vee Q$ and $\neg Q$, are true. According to the above table, the truth value of the conclusion, P, is TRUE in the second line. This shows that Example 1 is a valid argument!

Logical Equivalence

We say that two statements, P and Q, are *logically equivalent* when they ALWAYS have the same truth value no matter what and we denoted this as $P \equiv Q$.

Stop and Think

Why do we care about two statements being logically equivalent?

Let's say we have two logically equivalent statements, P and Q. If we know that P is true, then what do we know about Q? Since they always have the same truth value, we automatically know that Q is true as well. This is extremely useful when we want to find a simpler logical statement for complex statement.

Exercise 3

Show that $\neg(P \wedge Q)$ is equivalent to $\neg P \vee \neg Q$ by making a truth table.

Exercise 3 Solution

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Notice that the truth values of both statements are always equal. Hence, we know that they are logically equivalent.

Stop and Think

What English sentences do $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ represent in Example 1?

- $\neg(P \land Q)$: I will not both go to work tomorrow and today.
- $\neg P \lor \neg Q$: I will not go to work tomorrow or today. (Please keep in mind that $\lor$ in mathematics is inclusive.)

Notice that English sentences that are represented by $\neg(P \land Q)$ and $\neg P \lor \neg Q$ have the equivalent meaning.

Exercise 3 is called *De Morgan's law*. Here are other laws which state more logically equivalent statements:

1. De Morgan's laws
$$\neg(P \land Q) \equiv \neg P \lor \neg Q.$$
$$\neg(P \lor Q) \equiv \neg P \land \neg Q.$$
2. Commutative laws
$$P \land Q \equiv Q \land P.$$
$$P \lor Q \equiv Q \lor P.$$
3. Associative laws
$$P \land (Q \land R) \equiv (P \land Q) \land R.$$
$$P \lor (Q \lor R) \equiv (P \lor Q) \lor R.$$
4. Idempotent laws
$$P \land P \equiv P.$$
$$P \lor P \equiv P.$$
5. Distributive laws
$$P \land (Q \lor R) \equiv (P \land Q) \lor (P \land R).$$
$$P \lor (Q \land R) \equiv (P \lor Q) \land (P \lor R).$$
6. Absorption laws
$$P \lor (P \land Q) \equiv P.$$
$$P \land (P \lor Q) \equiv P.$$
7. Double Negation law
$$\neg\neg P \equiv P.$$

We can use the above laws to simplify complex logical statements. We can do so by finding a simpler equivalent logical statements. Here is an example.

Example 3

Use the laws above to find a simpler formula equivalent to $\neg(\neg P \vee Q) \vee (P \wedge \neg R)$.

Example 3 Solution

$$\begin{aligned}
\neg(\neg P \vee Q) \vee (P \wedge \neg R) &\equiv (\neg\neg P \wedge \neg Q) \vee (P \wedge \neg R) && \text{(De Morgan's law)}\\
&\equiv (P \wedge \neg Q) \vee (P \wedge \neg R) && \text{(Double Negation law)}\\
&\equiv P \wedge (\neg Q \vee \neg R) && \text{(Distributive law)}\\
&\equiv P \wedge \neg(Q \wedge R) && \text{(De Morgan's law)}
\end{aligned}$$

Logic Puzzles

Example 4

There are four people. They each have different colour house (they are in a row), nationality, favourite animal and sport. Which position the Blue house is in?

A. There are two houses between the person who likes Bowling and the person who likes Swimming.

B. There is one house between the Irish and the person who likes Handball on the left.

C. The second house is Black.

D. There is one house between the person who likes Horses and the Red house on the right.

E. The American lives directly to the left of the person who likes Turtles.

F. There are two houses between the person who likes Horses and the person who likes Butterflies on the right.

G. The person who likes Bowling lives somewhere to the right of the person who likes Tennis.

H. There is one house between the person who likes Handball and the White house on the right.

I. The British lives in the first house.

Taken from brainzilla (https://www.brainzilla.com/)

Example 4 Solution

	House #1	House #2	House #3	House #4
Colour				
Nationality				
Animal				
Sport				

1. Both C and I are simple clues which provide direct information about the colour of House #2 and the nationality of person in House #1, respectively.
2. Since there are four houses in total, F tells us that the person who likes Horses and the person who likes Butterflies must live in the houses at the end (House #1 and #4, respectively since the person who likes Butterflies is on the right).
3. D tells us that there is one house between the person who likes Horses (House #1) and the Red house on the right. Hence, the Red house must be House #3.
4. Similar to 2., by A, we know that the person who likes Bowling and the person who likes Swimming are at the end. However, we don't know which one is which this time. By G, we know that there is at least one house on the left side of the house of the person who likes Bowling. So the person who likes Bowling cannot be in House #1, hence, they must live in House #4 and the person who likes Swimming must be in House #1.
5. By E, we know that the American must live in **either** House #2 or #3. However, if they live in House #3, they would have been living in the house directly to the left of the person who likes Butterflies (House #4). Therefore, the American lives in House #2 and the person who likes Turtles lives in House #3.
6. We know that the Irish must live in **either** House #3 or #4. However, if they live in House #3, there is one house between them and the person who likes Swimming on the left which makes the clue B false. Therefore, the Irish must live in House #4 and the person who likes Handball must live in House #2.
7. By H, we know that House #4 must be White.

	House #1	House #2	House #3	House #4
Colour	**Blue**	Black	Red	White
Nationality	British	American		Irish
Animal	Horses		Turtles	Butterflies
Sport	Swimming	Handball		Bowling

Therefore, House #1 is the BLUE house!

Stop and Think

Was the word **either** in the solution above used as *exclusive or* OR *inclusive or*?

Notice how we used deductive reasoning to solve the logic puzzle. We assumed that all the clues (premises) are true to get to the conclusion "House #1 is blue."

Further Discussion - Paradoxes

A *paradox* is a logically self-contradictory statement. Here are a couple of paradoxes for you to discuss with your peers. Think about how they might be related to what we have learned so far.

Paradox 1

Socrates arrives at a bridge guarded by a powerful lord, Plato, and begs to be allowed to cross. Plato replies:

I swear that if the next utterance you make is true I shall let you cross,
but if it is false I shall throw you in the water.

Socrates replies:

You are going to throw me in the water.

If Plato does not throw him in the water, Socrates has spoken falsely and should be thrown in; but if he is thrown in, Socrates has spoken truly and should not be thrown in.

Taken from *Paradoxes from A to Z* by Michael Clark.

Paradox 2

If I say that I am lying, am I telling the truth? If I am, I am lying and so uttering a falsehood; but if I am not telling the truth I am lying, and so I *am* telling the truth. So my utterance is both true and false.

Taken from *Paradoxes from A to Z* by Michael Clark.